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BLACK-SCHOLES EQUATION

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Abstract. The purpose of this paper is to present approximation of C_0-sequentially equicontinuous semigroups on a sequentially complete locally convex space X.

1. Introduction

In 1973, Black and Scholes showed that under certain natural assumptions about the financial market, the price of a European option V, as a function of time τ and the current value of the underlying asset s, satisfies the final value problem

$$\begin{cases} \frac{\partial V}{\partial \tau} & = -\frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} - rs\frac{\partial V}{\partial s} + rV, \quad 0 \le \tau \le \bar{\tau},\ 0 \le s < \infty \\ & V(s,\ \bar{\tau}) = \bar{h}(s), \end{cases}$$

where σ is the volatility, r is the risk-free interest rate and $\bar{\tau}$ is the expiry date. By introducing new variables

$$t = \frac{1}{2}(\bar{\tau} - \tau)\sigma^2, \quad x = \ln s \quad \text{and} \quad u(x,\ \tau) = V(s,\ \tau),$$

we have the following initial value problem

$$\begin{cases} \frac{\partial u}{\partial t} & = \frac{\partial^2 u}{\partial x^2} + (\gamma - 1)\frac{\partial u}{\partial x} - \gamma u, \\ & u(x, 0) = h(x), \end{cases}$$

where $\gamma = 2r/\sigma^2 < 1$ is a constant and $h(x) = \bar{h}(e^x)$.

Rewriting this equation in terms of a differential operator A, we can interpret this equation as an abstract Cauchy problem $u'(t) = Au(t)$, $u(0) = f$. It is well-known that the solution of the abstract Cauchy problem for a linear operator $A : D(A) \to X$ on a Banach space X and

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an initial value $f \in D(A)$ is given by $u(t)S(t)f$ if A is the generator of a C_0 semigroup $\{S(t) : t \geq 0\}$. And thus the price oft he option is obtained from the semigroup. Therefore it is crucial to determine whether the Black-Scholes operator generates a C_0 semigroup or not.

In this paper we will show that the differential operator

$$\frac{d}{dx^2} + (\gamma - 1)\frac{d}{dx} - \gamma I$$

is a perturbation of the square of a generator of a C_0 group and it generates a C_0 semigroup on a suitable Banach space X and then we will present the approximation of this semigroup by discrete semigroups.

2. Approximation

We recall some definitions of the C_0 semigroup. For more information about the C_0 semigroup, see[P]. Let X be a Banach space.

Definition 2.1. A family $\{T(t) : t \geq 0\}$ of bounded linear operators from X into itself is called a C_0 semigroup on X if

(i) $T(0) = I$, the identity operator on X and $T(t + s) = T(t)T(s)$ for $t,\ s \geq 0$
(ii) $\lim_{t\to 0} T(t)x = x$ for all $x \in X$.

$\{T(t) : t \geq 0\}$ is called a C_0 semigroup of contractions if $\|T(t)\| \leq 1$ for all $t \geq 0$. If the properties (i) and (ii) hold for all $t \in \mathbf{R}$, we call $\{T(t) : t \in \mathbf{R}\}$ a C_0 group.

The generator of $\{T(t) : t \geq 0\}$ is the linear operator A, given by

$$Ax = \lim_{h\to 0} \frac{1}{h}(T(h)x - x)$$

with $D(A) = \{x \in X : \lim_{h\to 0} \frac{1}{h}(T(h)x - x) \in X\}$.

In order to trnsform Black-Scholes equation into the abstract Cauchy problem, consider the Banach space of continuous functions vanishing at infinity

$$X = C_0(\mathbf{R}) = \{f \in C(\mathbf{R}) : \lim_{|x|\to\infty} f(x) = 0\}$$

with the usual supremum norm $\|f\| = \sup_{x\in\mathbf{R}} |f(x)|$.

Let $(T(t)f)(x) = f(t + x)$ for $t \in \mathbf{R}$, $f \in X$ and $x \in \mathbf{R}$. Then it is not difficult to show that $\{T(t) : t \in \mathbf{R}\}$ is a C_0 group on X. Let B be its generator. Then for $f \in D(B)$ and $x \in \mathbf{R}$, we have

$$\begin{aligned}(Bf)(x) &= \lim_{h\to 0}\frac{1}{h}(T(t)f(x) - f(x)) \\ &= \lim_{h\to 0}\frac{1}{h}(f(x+h) - f(x)) = f'(x).\end{aligned}$$

So f is differentiable and $Bf \in X$, that is,

$$D(B) \subseteq C_0^1(\mathbf{R}) = \{f \in C^1(\mathbf{R}) : f,\ f' \in X\}.$$

Conversely, let $f \in C_0^1(\mathbf{R})$. For $x \in \mathbf{R}$,

$$\begin{aligned}&|\frac{1}{h}(T(h)f(x) - f(x)) - f'(x)| \\ &\quad = |\frac{1}{h}(f(x+h) - f(x)) - f'(x)| = |\frac{1}{h}\int_0^h (f'(x+\tau) - f'(x))d\tau| \\ &\quad \le \sup_{0\le|\tau|\le|h|} |f'(x+\tau) - f'(x)| \ \to\ 0,\end{aligned}$$

as $h \to 0$ uniformly in $x \in \mathbf{R}$, since $f' \in C_0(\mathbf{R})$ is uniformly continuous. Therefore $B = d/dx$ is a generator of a C_0 semigroup $\{T(t) : t \in \mathbf{R}\}$ of contractions.

THEOREM 2.2. *Let A be the operator defined by*

$$A = B^2 + (\gamma - 1)B = \frac{d^2}{dx^2} + (\gamma - 1)\frac{d}{dx}$$

with $D(A) = D(B^2) = \{f \in D(B) : Bf \in D(B)\}$. Then A is the generator of a contraction C_0 semigroup $\{S(t) : t \ge 0\}$ on X.

Proof. Note that B is the generator of a contraction C_0 group $\{T_2(t) : t \in \mathbf{R}\}$ on X, where $T_1(t)f(x) = f(x+t)$ for $x \in \mathbf{R}$. By Corollary 3.7.5 in [A], B^2 is the generator of a bounded holomorphic C_0 semigroups $\{T_1(t) : t \ge 0\}$ on X, which is given by

$$T_2(t)f = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{s^2}{4t}} T_1(s)f ds.$$

For $t \ge 0$, we have

$$\|T_2(t)f\| \le \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{s^2}{4t}} \|T_1(s)f\| ds \le \|f\|.$$

So $\{T_2(t) : t \ge 0\}$ is a contraction C_0 semigroups on X.

By Lemma 2.8 of chapter 1 in [P], for $f \in D(B^2)$ and $\varepsilon > 0$

$$\begin{aligned} ||Bf|| &\leq 2(\|B^2 f\|\|f\|)^{1/2} = 2(\varepsilon\|B^2 f\|)^{1/2}(\frac{1}{\varepsilon}\|f\|)^{1/2} \\ &\leq \varepsilon\|B^2 f\| + \frac{1}{\varepsilon}\|f\| \end{aligned}$$

By Corollary 3.3 of chapter 3 in [P], $B^2 + (\gamma - 1)B$ is the generator of a contraction C_0 semigroup $\{S(t) : t \geq 0\}$. □

Remark 2.3. $A - \gamma I = d^2/dx^2 + (\gamma - 1)d/dx - \gamma I$ generates a contraction C_0 semigroup $\{e^{-\gamma t}S(t) : t \geq 0\}$ on X.

Next we will present the approximation of the C_0 semigroup $\{S(t) : t \geq 0\}$ generated by A.

Let X_n be the space of all bounded real sequences $\{c_k\}_{-\infty}^{\infty}$ satisfying $\lim_{|k|\to\infty} c_k = 0$ with the usual supremum norm. Define linear operators $P_n : X \to X_n$ and $E_n : X_n \to X$ by

$$P_n f(x) = \{f(k/n)\}_{k=-\infty}^{\infty} \quad \text{and} \quad E_n(\{c_k\}_{-\infty}^{\infty}) = g(x),$$

where $g(k/n) = c_k$ and $g(x)$ is linear between two consecutive points $k/n$and $(k+1)/n$. Then $\{E_n\}$ and $\{P_n\}$ satisfy the assumption 6.1 in chapter 3 in [P].

Theorem 2.4. *Let A be the generator of a C_0 semigroup $\{S(t) : t \geq 0\}$ in Theorem. We define a linear operator $F(\rho_n) : X_n \to X_n$ by*

$$\begin{aligned} F(\rho_n)(c) &= \{(\frac{1}{2} - 2n^2\rho_n)c_k + n^2\rho_n(c_{k+1} + c_{k-1}) \\ &\quad +(\frac{1}{2} + (\gamma - 1)n\rho_n)c_k - (\gamma - 1)n\rho_n c_{k-1}\}_{k=-\infty}^{\infty} \end{aligned}$$

for $c = \{c_k\}_{k=-\infty}^{\infty}$ in X_n and some $\rho_n > 0$ such that $4n^2\rho_n < 1$. Then we have

$$\lim_{n\to\infty} \|F(\rho_n)^{k_n} P_n f - P_n S(t)f\| = 0$$

for $f \in X$ and a sequence $\{k_n\}$ of positive integers such that $\lim_{n\to\infty} k_n\rho_n = t$.

Proof. First we will show that $F(\rho_n)$ is a contraction. For $c = \{c_k\}_{k=-\infty}^{\infty}$ in X_n

$$\begin{aligned} \|F(\rho_n)(c)\| &\leq \sup_k \{(\frac{1}{2} - 2n^2\rho_n)|c_k| + n^2\rho_n(|c_{k+1}| + |c_{k-1}|) \\ &\quad +(\frac{1}{2} + (\gamma-1)n\rho_n|c_k| + (1-\gamma)n\rho_n|c_{k-1}|\} \\ &\leq ((\frac{1}{2} - 2n^2\rho_n) + 2n^2\rho_n + (\frac{1}{2} + (\gamma-1)n\rho_n) + (1-\gamma)n\rho_n)\|c\| \\ &= \|c\|. \end{aligned}$$

So $F(\rho_n)$ is a contraction. And we have for $f \in D(A)$

$$\begin{aligned} &\|\frac{1}{\rho_n}(F(\rho_n) - I)P_n f - P_n A f\| \\ &\leq \sup |n^2(f(\frac{k+1}{n}) - 2f(\frac{k}{n}) + f(\frac{k-1}{n}) - f''(\frac{k}{n})| \\ &\quad + \sup |n(f(\frac{k+1}{n}) - f(\frac{k}{n}) - f'(\frac{k}{n})|. \end{aligned}$$

Since $D(A)$ is dense in X, $f'(x)$ and $f''(x)$ are uniformly continuous on $\mathbf{R}$. By Theorem 6.7 of chapter 3 in [P], the result follows. □

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